Goldstone Bosons in the Gaussian Approximation

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Abstract

The $O(N)$ symmetric scalar quantum field theory with $\lambda\Phi^4$ interaction is discussed in the Gaussian approximation. It is shown that the Goldstone theorem is fulfilled for arbitrary N.

1 Introduction

The theory of a real scalar field in n-dimensional Euclidean space-time with a classical action given by

$$S[\Phi] = \int [\frac{1}{2}\Phi(x)(-\partial^2 + m^2)\Phi(x) + \lambda(\Phi^2(x))^2]\, d^n x, \tag{1}$$

is the most mysterious part of the standard model. Althought experimentaly not observed, the scalar Higgs field with $m^2 < 0$ and internal $O(4)$ symmetry is necessary to give masses to interaction bosons in the Weinberg-Salam model of weak interactions without spoiling renormalizability. Moreover, the renormalized $\lambda\Phi^4$ theory has been almost rigorously proved [1] to be noninteracting, in contradiction to the perturbative renormalization, which can be performed order by order without any signal of triviality. Triviality shows up in the leading order of the $\frac{1}{N}$ expansion [2] in the $O(N)$ symmetric

theory of $N-$component scalar field, $\Phi(x) = (\Phi_1(x), ..., \Phi_N(x))$. Other non-perturbative methods, like the Gaussian [3] and post-Gaussian [4, 5] approximations, have been therefore applied to study renormalization of the scalar theory in the case when the number of field components is not large. However, a serious drawback of the Gaussian approximation for N-component field was an observation [6, 7] that the Goldstone theorem seemed not to be respected exactly, only in the limit of $N \to \infty$ did the would-be Goldstone bosons become massless. Here we show that this statement is not true, and is due to a faulty interpretation. In fact, the Gaussian approximation of the $O(N)$-symmetric theory yields one massive particle and $(N-1)$ massless Goldstone bosons, in agreement with the exact result of Goldstone theorem [8]. There is another claim of existence of Goldstone bosons in the Gaussian approximation by Dmitrasinovic at al. [9], who found a pole in the four-point Green function of the $O(2)$-symmetric theory, which was interpreted as a bound state of two massive elementary excitations. In our work we show that massless bosons appear in the Gaussian approximation as elementary fields which are eigenvectors of the one-particle propagator matrix (two-point Green function).

It is convenient to formulate the approximation method for the effective action, $\Gamma[\varphi]$, since all the Green's functions can be obtained in a consistent way, through differentiation of the approximate expression. A full (inverse) propagator, required for one-particle states analysis, is given by the second derivative of the effective action at $\varphi(x) = \phi_0$. The vacuum expectation value of the scalar field, ϕ_0, can be obtained as a stationary point of the effective potential, $V(\phi) = -\frac{1}{\int d^n x}\, \Gamma[\varphi]|_{\varphi(x)=\phi=const}$.

We shall calculate the effective action, using the optimized expansion (OE) [4]. The method consists in modifying the classical action of a scalar field (1) to the form

$$S_\epsilon[\Phi, G] = \int \frac{1}{2}\Phi(x) G^{-1}(x,y)\Phi(y)\, d^n x d^n y$$
$$+\epsilon \left[\int \frac{1}{2}\Phi(x)[-\partial^2 + m^2)\delta(x-y) - G^{-1}(x,y)]\Phi(y)\, d^n x d^n y + \int \lambda (\Phi^2(x))^2\, d^n x \right], \quad (2)$$

with an arbitrary free propagator $G(x, y)$. The effective action, as a series in an artificial parameter ϵ, can be obtained as a sum of vacuum one-particle-irreducible diagrams with Feynman rules of the modified theory. The given order expression for the effective action is optimized, choosing $G(x, y)$ which

fulfills the gap equation

$$\frac{\delta\Gamma_n}{\delta G^{-1}(x,y)} = 0, \tag{3}$$

to make the dependence on the unphysical field as weak as possible.

It has been shown that in the first order of the OE for one-component field, the inverse of a free propagator can be taken in the form

$$\Gamma(x,y) = G^{-1}(x,y) = (-\partial^2 + \Omega^2(x))\delta(x-y), \tag{4}$$

and the Gaussian effective action (GEA) is obtained [4]. The effective potential, derived from the GEA for a constant background $\varphi(x) = \phi$, coincides with the Gaussian effective potential (GEP) [3], obtained before by applying the variational method with Gaussian trial functionals to the functional Schrödinger equation.

Here we shall calculate the effective action for N-component field to the first order of the OE. In this case, the inverse of a trial propagator can be chosen in the form of a symmetric matrix

$$\begin{aligned} \Gamma_{i,i}(x,y) &= (-\partial^2 + M_i^2(x))\delta(x-y) \\ \Gamma_{i,j}(x,y) &= \Gamma_{j,i}(x,y) = M_{ij}^2(x)\delta(x-y) \end{aligned} \tag{5}$$

where the functions $M_i^2(x)$ and $M_{ij}^2(x)$ are variational parameters. The calculation of the effective action can be simplified, using the observation of Stevenson at al. [7] that for an $O(N)$ symmetric theory only the shift $\varphi(x) = (\varphi_1(x), ..., \varphi_N(x))$ of the field sets a direction in the $O(N)$ space. Thus, the eigendirection of a free propagator matrix will be radial and transverse, and the variational parameters for the transverse fields should be equal, because of the remaining $O(N-1)$ symmetry. In the coordinate system, in which the shift φ points in the $i = 1$ direction, the (inverse) trial propagator can be chosen in the form of a diagonal matrix with

$$\begin{aligned} \Gamma_{11}(x,y) &= G^{-1}(x,y) = (-\partial^2 + \Omega^2(x))\delta(x-y) \\ \Gamma_{ii}(x,y) &= g^{-1}(x,y) = (-\partial^2 + \omega^2(x))\delta(x-y) \qquad \text{for } i \neq 1, \end{aligned} \tag{6}$$

and the effective action in the first order of the OE is obtained in the form

$$\Gamma[\varphi] = -\int [\frac{1}{2}\varphi(x)(-\partial^2 + m^2)\varphi(x) + \lambda(\varphi^2(x))^2]\, d^n x - \frac{1}{2} Tr Ln G^{-1}$$

$$
\begin{aligned}
&- \frac{N-1}{2} Tr Lng^{-1} + \frac{1}{2}\int (\Omega^2(x) - m^2 - 12\lambda\varphi^2(x))G(x,x)\, d^n x \\
&+ \frac{(N-1)}{2}\int (\omega^2(x) - m^2 - 4\lambda\varphi^2(x))g(x,x)\, d^n x - 3\lambda \int G^2(x,x)\, d^n x \\
&- (N^2-1)\lambda \int g^2(x,x)\, d^n x - 2(N-1)\lambda \int G(x,x)g(x,x)\, d^n x. \qquad (7)
\end{aligned}
$$

Requiring the effective action to be stationarity with respect to small changes of variational parameters

$$
\frac{\delta\Gamma}{\delta\Omega^2} = \frac{\delta\Gamma}{\delta\omega^2} = 0, \tag{8}
$$

results in gap equations

$$
\begin{aligned}
\Omega^2(x) - m^2 - 12\lambda\varphi^2(x) - 12\lambda G(x,x) - 4(N-1)\lambda g(x,x) &= 0 \\
\omega^2(x) - m^2 - 4\lambda\varphi^2(x) - 4\lambda G(x,x) - 4(N+1)\lambda g(x,x) &= 0 \qquad (9)
\end{aligned}
$$

which determine the functionals $\Omega[\varphi]$ and $\omega[\varphi]$. When limited to a constant background $\phi = (\phi_1, ..., \phi_N)$, the the GEA for N-component field gives the effective potential

$$
\begin{aligned}
V(\phi) &= \frac{m^2}{2}\phi^2 + \lambda(\phi^2)^2 + I_1(\Omega) + (N-1)I_1(\omega) + \frac{1}{2}(m^2 - \Omega^2 + 12\lambda\phi^2)I_0(\Omega) \\
&+ \frac{N-1}{2}(m^2 - \omega^2 + 4\lambda\phi^2)I_0(\omega) + 3\lambda I_0(\Omega)^2 + (N^2-1)\lambda I_0(\omega)^2 \\
&+ 2(N-1)\lambda I_0(\Omega)I_0(\omega) \qquad (10)
\end{aligned}
$$

with the functions $\Omega(\phi)$ and $\omega(\phi)$ determined by algebraic equations

$$
\begin{aligned}
\Omega^2 - m^2 - 12\lambda\phi^2 - 12\lambda I_0(\Omega) - 4\lambda(N-1)I_0(\omega) &= 0, \\
\omega^2 - m^2 - 4\lambda\phi^2 - 4\lambda I_0(\Omega) - 4(N+1)\lambda I_0(\omega) &= 0, \qquad (11)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(\Omega) &= \frac{1}{2}\int \frac{d^n p}{(2\pi)^n} \ln(p^2 + \Omega^2) \\
I_0(\Omega) &= \int \frac{d^n p}{(2\pi)^n}\frac{1}{p^2 + \Omega^2}. \qquad (12)
\end{aligned}
$$

The same result for the $O(N)$ symmetric GEP was obtained before in the Schrödinger approach [7]. In the OE, a generalisation of the GEP to space-time dependent fields, the GEA (7), has been obtained. It enables us to derive not only the effective potential, but also one-particle-irreducible Green's functions at arbitrary external momenta in the Gaussian approximation.

The minimum of the GEP is at ϕ_0 fulfilling

$$\frac{\partial V}{\partial \phi_i} = (m^2 + 4\lambda\phi^2 + 12\lambda I_0(\Omega) + 4(N-1)\lambda I_0(\omega))\phi_i = 0; \tag{13}$$

therefore, in the unsymmetric minimum we have

$$B = m^2 + 4\lambda\phi^2 + 12\lambda I_0(\Omega) + 4(N-1)\lambda I_0(\omega) = 0. \tag{14}$$

In the GEP analysis for $N = 2$, it was pointed out by Brihaye and Consoli [6] that $\omega[\phi_0]$ is not equal to zero, which was interpreted as a violation of Goldstone theorem in the Gaussian approximation. For the same reason, Stevenson, Allès and Tarrach [7] admitted that also for a general N the Gaussian approximation does not respect the Goldstone theorem. We would like to point out that this conclusion is unjustified, for Ω and ω are only variational parameters in the free propagator, and do not correspond to physical masses of scalar particles. The physical masses have to be determined as poles of a full propagator in the discussed approximation. The inverse of that propagator can be obtained as a second derivative of the GEA (7) with an implicit dependence, $\Omega^2[bf\varphi]$ and $\omega^2[\varphi]$, taken into account by differentiation of the gap equations (9). Upon performing the Fourier transform, the two-point vertex is calculated to be

$$\begin{aligned}
\Gamma_{11}(p) &= \left.\frac{\widehat{\delta^2\Gamma}}{\delta\varphi_1^2}\right|_{\varphi(x)=\phi_0} = p^2 + m^2 + 4\lambda\phi^2 + 12\lambda I_0(\Omega) + 4(N-1)\lambda I_0(\omega) + 8\lambda\phi_1^2 A(p) \\
\Gamma_{ii}(p) &= \left.\frac{\widehat{\delta^2\Gamma}}{\delta\varphi_i^2}\right|_{\varphi(x)=\phi_0} = p^2 + m^2 + 4\lambda\phi^2 + 12\lambda I_0(\Omega) + 4(N-1)\lambda I_0(\omega) + 8\lambda\phi_i^2 A(p) \\
\Gamma_{ij}(p) &= \Gamma_{ji}(p) = \frac{1}{2}\left.\frac{\widehat{\delta^2\Gamma}}{\delta\varphi_i\delta\varphi_j}\right|_{\varphi(x)=\phi_0} = 8\lambda\phi_i\phi_j A(p),
\end{aligned} \tag{15}$$

where

$$A(p) = 1 - \frac{18\lambda I_{-1}(\Omega,p) + 2\lambda(N-1)I_{-1}(\omega,p) + 24\lambda^2(N+2)I_{-1}(\Omega,p)I_{-1}(\omega,p)}{1 + 6\lambda I_{-1}(\Omega,p) + 2\lambda(N+1)I_{-1}(\omega,p) + 32\lambda^2(N+2)I_{-1}(\Omega,p)I_{-1}(\omega,p)}. \tag{16}$$

and

$$I_{-1}(\Omega, p) = 2 \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 + \Omega^2)((p+q)^2 + \Omega^2)}. \tag{17}$$

Upon diagonalization of the matrix (15) we obtain an inverse propagator $\gamma_1(p) = p^2 + B + 8\lambda A(p)\phi_0^2$, which corresponds to massive particle, and $(N-1)$ inverse propagators $\gamma_i(p) = p^2 + B$ of Goldstone bosons, since B=0 in the unsymmetric minimum (14). Therefore, for any N the Gaussian approximation of the $O(N)$ symmetric theory does fully respect Goldstone theorem at the unrenormalized level.

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